Multivariable calculus

Now we have a good intuition for how these different types of waves behave. How could we describe this behavior mathematically? What would we need to know?

→ shape of the wave
ex. water: height of water

Consider water waves. We want to know the height of the water, but on what does the height depend?

→ position (lets consider one direction, call it x)
→ time (lets call this t)

So we know that the height h depends on the position x and the time t. We write this as

$$\underset{\text{height}}{h}\underbrace{(x,t)}_{\text{depends on two different things, } x \text{ and } t}$$

What is this an example of?

→ function

How many of you have seen functions before, or know what functions are?

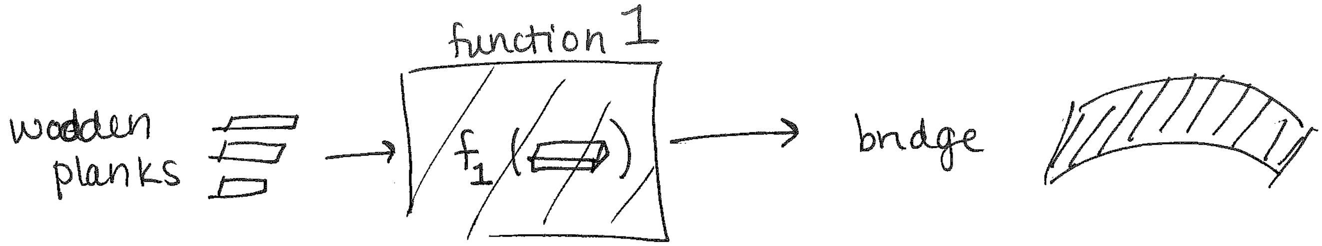
Multivariable calculus

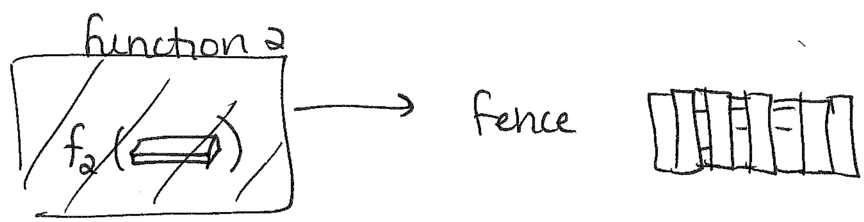
Aside: functions

A function is something that takes one object and turns it into another object. For example:

Imagine you have many planks of wood. You could do many things with that wood: build a bridge, make a fence, make a wall ...

A function is something that takes your wood and builds something with it. Each different function builds a different thing:

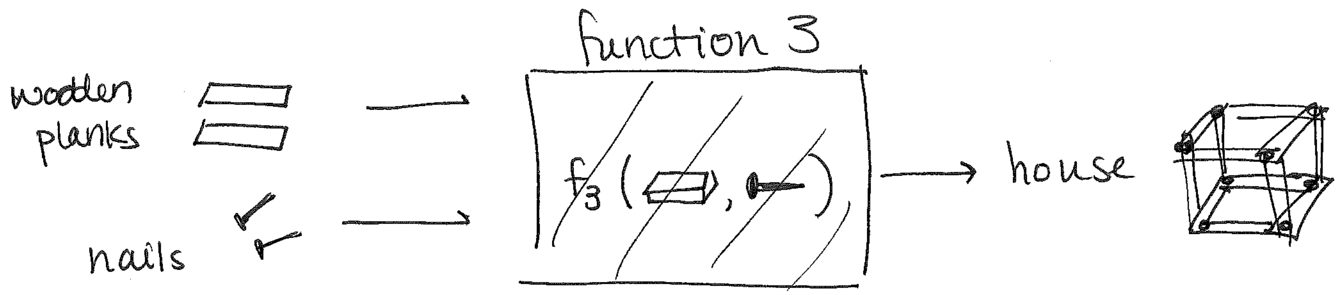




in each case, you start with wooden planks, but each function gives you a different output.

Now, imagine that you have both wooden planks and nails. Now you can build many more things, because you can hook the planks together in many ways

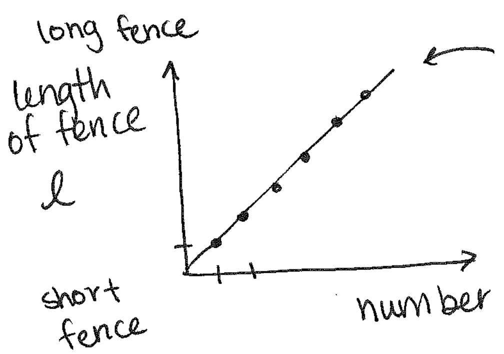
Multivariable calculus



the building materials (planks, nails) are called the <u>independent</u> variables. They are called "independent" b/c you can control how many of them you put into your function (how many you give to your builder)

The output (house, fence, bridge) is called the <u>dependent</u> variable. It is called "dependent") because it <u>depends</u> on how much you input, ie the size of the house depends on both how many planks and how many nails you have.

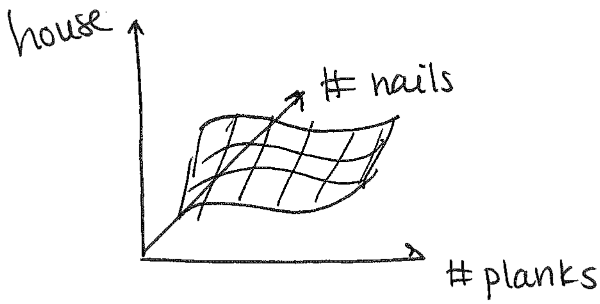
Let's try to plot some of these functions:



← function $\ell(\#)$ is just a straight line. The more planks you give your builder, the longer the fence he can build

Multivariable calculus

With both planks and nails, your graph looks different:



← looks like a floating carpet instead of floating string

In principle, your function could have as many inputs as you want, and they could be anything you want:

f(), g(, ), ...

where f and g are arbitrary names of the functions.

~~For~~ By convention, we often call position and time:

x - horizontal

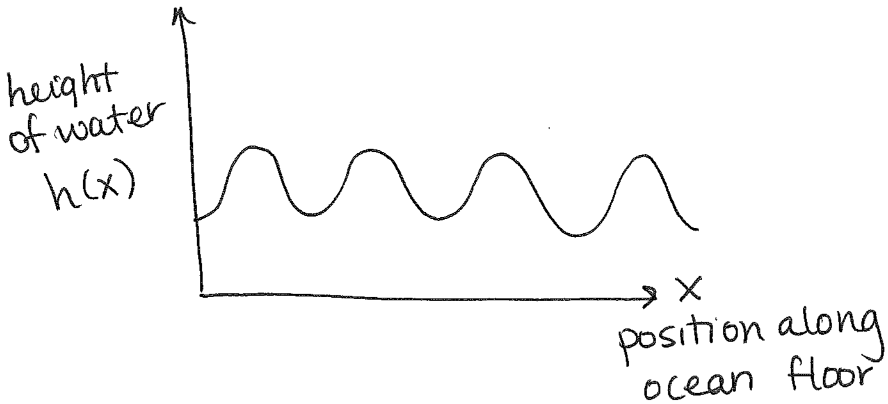
y - vertical

t - time

but these are just labels, we could name them whatever we want

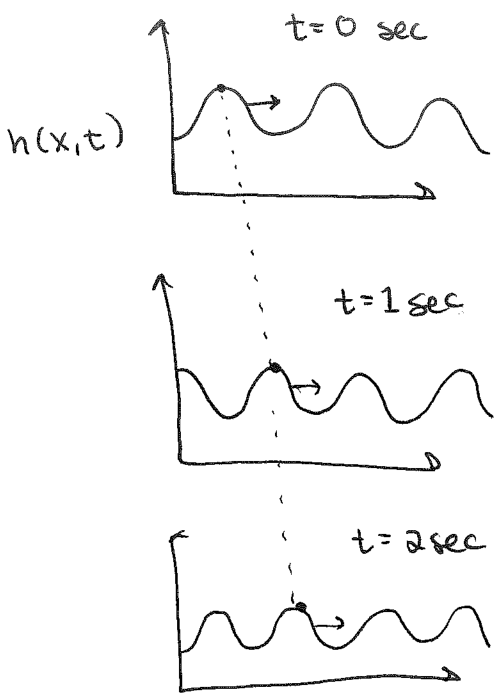
Multivariable calculus

Now, back to describing our water wave. Lets draw a picture of the wave:



h(x) : we ~~say~~ give a value of x, and our function h(x) tells us the height of the water at that position

Now, we know our wave is also changing in time



Now, we want to tell our function a specific position and a specific time, and have it tell us the height of the water at that value of x, t (ie how high ~~is~~ will the water be in Tahiti (x) in 2 days (t)?)

our full function is h(x,t)

if we can define h(x,t), then we know the height of the water at <u>every</u> position at <u>every</u> time?

For a complicated system, like the ocean (where there is wind variation, etc), ~~this~~ finding h(x,t) is very difficult. But for a small system (like a tank of water or wave on a string), this is doable.

Multivariable calculus

How do we figure out what $h(x,t)$ is?

→ make a guess?

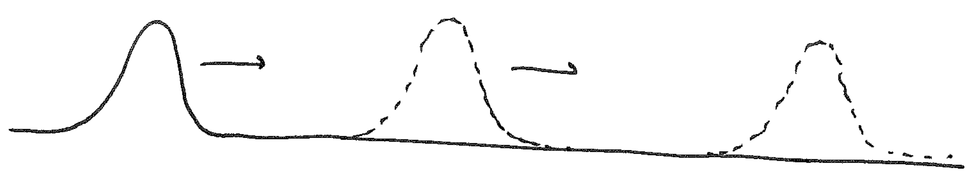
→ take many pictures of the wave + try to deduce the shape?

→ think about the physical laws that govern the wave motion?

Let's try option 3. Think about a wave on a string (a single peak traveling down the length of the string)

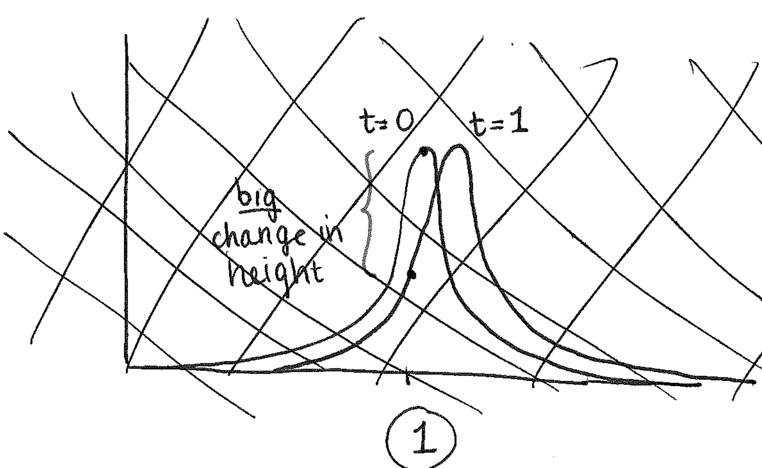
What happens as the wave travels down the string?

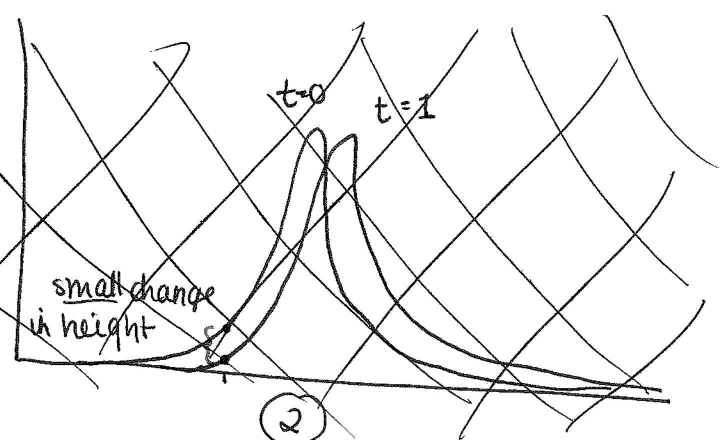
→ the shape stays the same



We want a mathematical description that captures this behavior

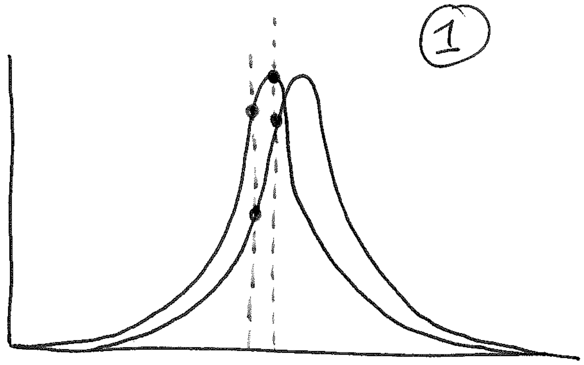
Let's focus on a single point along the string and look at how the height at that point changes





Multivariable calculus

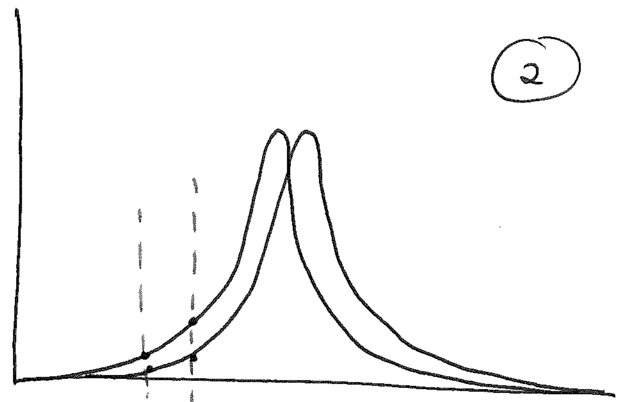
1



green - big change in height

red - ~~big~~ small change in height

2



green - small change in heigh

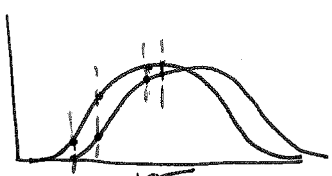
red - big change in height

change in height ~~is increasing quickly~~ changes a lot / quickly

change in height ~~is decreasing slowly~~ changes a little / slowly

What's the difference between 1 and 2 ?

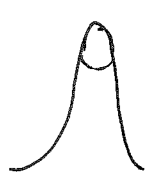
→ height? consider



changes a lot

changes a little

→ curvature. When the wave is more curved (higher curvature) there is a bigger change in the change in height.



higher curvature
smaller circle

lower curvature
bigger circle

Multivariable calculus

high curvature ⟷ rapid change in change in height

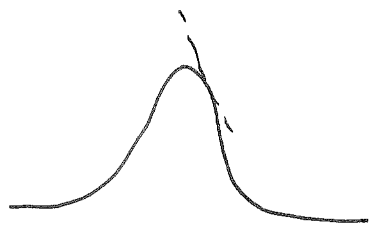
changes in height with respect to position

changes in height with respect to time

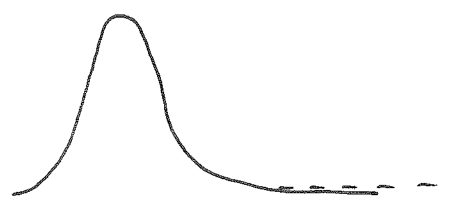
⟶ acceleration

What is curvature? How do we define it mathematically?

First, look at what we call slope - change in height wrt position.

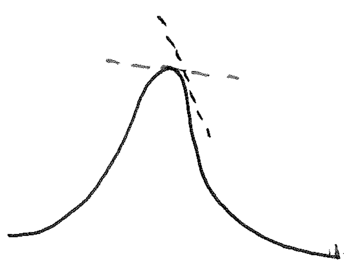


big slope — changing a lot

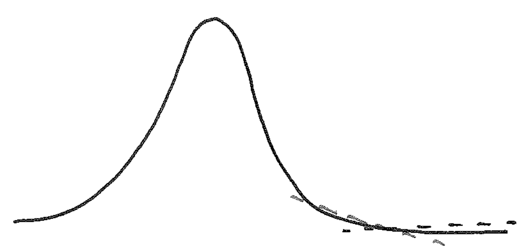


small / zero slope — changing a little or not at all

Now, what happens to the slope when there is high/low curvature?



high curvature - slope is changing a lot



low curvature - slope is changing a little.

* remember - slope is change in height wrt position

Multivariable calculus

high curvature $\longleftrightarrow$ high acceleration

change in change in height wrt position $\longleftrightarrow$ change in change in height wrt time

How do we mathematically describe change?

$\longrightarrow$ derivative

$\longrightarrow$ wrt position $\frac{d}{dx}$

$\longrightarrow$ wrt time $\frac{d}{dt}$

Change in change?

$\longrightarrow$ second derivative

$\longrightarrow$ wrt position $\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx} f(x)\right)$

$\longrightarrow$ wrt time $\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt} f(t)\right)$

Now, if we have a function of two variable (x,t), we ~~write derivatives~~ have to use partial derivatives - ~~this just tells us that~~

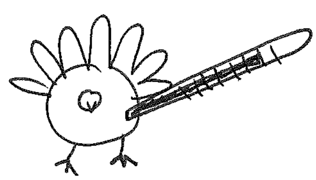
$\frac{d}{dx} \longrightarrow \frac{\partial}{\partial x}$ only take derivative wrt position, not wrt time, position + time, etc.

$\frac{d^2}{dx^2} \longrightarrow \frac{\partial^2}{\partial x^2}$

Multivariable calculus

You can think of a derivative, $\frac{d}{dx}$, as a probe, like a thermometer:

If you have a turkey in the oven, and you want to know the temp, you stick in a thermometer. The thermometer takes the turkey + tells you the temp:

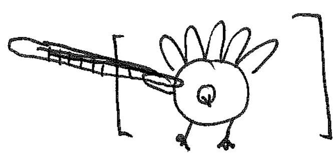


therm. – measure temp.

turkey

Similarly, we have a function (turkey). We want to know the slope (temp), so we use a derivative (thermometer).

$$\frac{d}{dx}\left[\ f(x)\ \right] = \text{slope}$$



$= 400°$!

The second derivative (~~curvature, acceleration~~) is a different probe that gives you the curvature or acceleration

Multivariable calculus

change in change in height wrt position $\longleftrightarrow$ change in change in height wrt time

$$\frac{\partial^2}{\partial t^2} h(x,t) \longleftrightarrow \frac{\partial^2}{\partial x^2} h(x,t)$$

We have just derived the wave eqn!

$$\frac{\partial^2}{\partial t^2} h(x,t) = c\frac{\partial^2}{\partial x^2} h(x,t) \qquad c = \text{constant}$$

This equation describes many types of waves - water waves, sound waves, waves on a string...

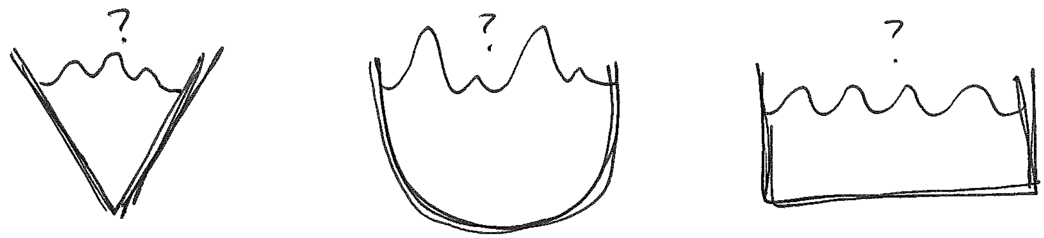
The wave eqn. is based on /derived from the physical behavior of waves. This equation is a tool that helps us find an equation for $h(x,t)$, ie it helps us figure out whether a wave looks like

We need one additional tool to find $h(x,t)$ — we need to know what is happening at the boundaries (b.c.'s)

$\longrightarrow$ We need to know what the edges of the string ~~on~~ look like, what the water tank looks like

Multivariable calculus

For example, water waves look different in different kinds of tanks:



(and remember that waves on a string w/ 2 fixed ends looked different than those w/ one fixed end)

If we want to ~~the~~ find $h(x,t)$ we need two things:

1. wave equation
2. boundary conditions